AD-A211 867

CERTAINTY EQUIVALENCE AND INEQUIVALENCE FOR PRICES

by

Kenneth J. Arrow

Technical Report No. 538

December, 1988

DTIC

1

CERTAINTY EQUIVALENCE AND INEQUIVALENCE FOR PRICES

by

Kenneth J. Arrow

Technical Report No. 538

December, 1988

PREPARED UNDER THE
UNITED STATES OFFICE OF NAVAL RESEARCH
ONR-N00014-86-K-0216

THE ECONOMICS SERIES

INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
Fourth Floor, Encina Hall
Stanford University
Stanford, California 94305

CERTAINTY EQUIVALENCE AND INEQUIVALENCE FOR PRICES

Kenneth J. Arrow

1. INTRODUCTION

A basis for the analysis of economic behavior under uncertainty has existed ever since Daniel Bernoulli's famous paper [1738]. Indeed, Bernoulli applied his expected-utility theory to explaining the demand for marine insurance, the problem, of course, being to explain positive demand for a risk with negative expected value. Bernoulli saw clearly that both the <u>Gedanken</u> evidence of the St. Petersburg paradox and the real-world purchase of insurance were simply statements that the certainty-equivalent of a risk was not its expected value; his clear analysis led him to the synthesis of an alternative theory of behavior.

Two hundred years passed before Marschak [1938, 1949], Markowitz [1952], Allais [1952], Arrow [1953], Hicks [1931, 1962], Tobin [1958], and many others began to undertake the task of a systematic analysis of specific economic actions, especially investments and the holding of assets, on the basis of an explicit theory of behavior under uncertainty. Most but not all of these papers used the Bernoulli model, attention to which was redirected by its derivation from an axiomatic basis by von Neumann and Morgenstern [1947, Appendix II] (Ramsey's earlier similar axiomatization [1931] had passed

*This work was supported by Office of Naval Research Grant N00014-86-K-0216 at the Institute for Mathematical Studies in the Social Sciences, Stanford University, Stanford, California

This paper originally was prepared for International Economic Association conference, "<u>Value and Capital After Fifty Years</u>," Bologna, Italy, 3-5 September 1988.

unnoticed, like the rest of his economic writings.) The lag is somewhat surprising. The classical economists could not easily take Bernoulli's ideas into account, but they fitted very naturally into the neoclassical framework. Indeed, Jevons [1957, pp. 72-74; first edition, 1871] stated the expected-utility framework, though without reference to Bernoulli; and Marshall [1948, pp. 135, 843; first edition, 1890] cites Bernoulli explicitly. But neither evidently understood well how to formulate economic choice under uncertainty sharply enough to use expected-utility maximization as a tool.

The general lines which such a theory would have to take were already foreseen in an early paper of Hicks [1931]; a probability distribution over outcomes is itself a new kind of commodity, which has to be treated like other commodities. But no specific derivation of demand or supply for a commodity was undertaken in that paper. Marschak's [1938] derivation of demand for money and other assets followed in principle Hicks's point of view, though Marschak made the preferences over probability distributions depend on their first two or three moments only.

Thus, when Hicks began the presentation and elaboration of general equilibrium theory [1939] which we celebrate here, there was no adequate way of representing uncertainty. But to Hicks the presence of uncertainty was pervasive and essential to an understanding of the workings of the economy over time. Among the many great contributions of Value and Capital, to my mind, the greatest of all was the representation of future goods symmetrically with present ones. At one stroke, all the conceptual mysteries of capital theory and the confusions about steady states were wiped out. But no scholar as serious as Hicks would want to deny that we are less than certain about the future. (There is a quotation, which has had several attributions: "Prediction is always difficult, especially of the future.") Hence, some attempt to introduce uncertainty was essential. Irving Fisher [1930] in a similar but much simpler model also introduced uncertainty as an additional complication, but without any consistent analytic method.

It was to meet this gap that Hicks introduced the concept which has come be called <u>certainty-equivalence</u> (see Hicks [1939], pp. 125-126). It is important to make clear that Hicks was concerned with uncertainty about <u>prices</u>, not about quantities. In a "spot" economy, a firm chooses a production plan from a known transformation surface but is uncertain about some of the prices it would face. A typical but by no means most general case would be that of a firm choosing inputs today, under known prices, which lead to outputs tomorrow at prices unknown today. Then Hicks's hypothesis is that there is a vector of tomorrow's prices which, if known for certain, would lead to the same choice of production plan. These are the certainty-equivalent prices. He further formulates more specific hypotheses about the certainty-equivalent prices of future outputs; because of risk aversion, they will be below the expected value of future prices.

More generally, the production plan adopted today may require future inputs as well as outputs. It is to Hicks that we owe the symmetry of treatment of inputs and outputs; one is related to the other merely by change of sign. The certainty-equivalent price vector includes components for future inputs as well as outputs and is defined the same way. However, risk aversion now is taken to imply that the certainty-equivalent price of a future input is above the expected price.

(In Hicks's world, there are markets for bonds of all maturities. The future prices referred to above are to be interpreted, of course, as discounted prices. I will maintain this interpretation without further comment.)

The notion of certainty equivalence is extended by Hicks to the household as well as the firm. In this paper, however, I will deal only with certainty equivalence for the firm.

As Hicks ([1946], p. 134) notes, different firms may have different certainty equivalents (abbreviated here as CEs) for the same price. Even if they share probability beliefs about prices, Hicks postulates, as already seen, that a buyer will have a higher certainty equivalent than a seller. Hence, the markets will not be in full equilibrium, as Hicks strongly points out.

H. Theil [1954] showed that for a certain class of decision problems and for quadratic utility functions, the rational individual acts as if the expected value were the CE. One also hears frequently the claim that CEs exist when the utility function is exponential (constant absolute risk aversion) and income, for any given choice of decision variables, is a linear function of a random variable with a multivariate normal distribution. It sounds, then, as if the existence of CEs depends on particular assumptions on the utility function and the distribution of the unknown state of nature. I will argue that this reasoning is incorrect; in fact, for the firm's decision problem as formulated in Value and Capital (V&C, as it will be hereafter referred to), there always exists a certainty equivalent.

However, the V&C model of the firm's behavior under uncertainty is not adequate to represent the possiblity of flexible planning, as explicitly noted (Hicks [1946], p. 124) and as developed in other work, most notably on liquidity preference. When production planning can be flexible, one would have to distinguish between *ex ante* and *ex post* CEs. But there need not exist any CE prices in either sense.

2. INFLEXIBLE PRODUCTION PLANS AND CERTAINTY EQUIVALENTS

In V&C, the firm chooses a production vector, y, from a transformation surface, defined by $T(y) = 0$. (Vectors will be taken as column vectors, unless marked with a prime, denoting transposition.) In most later work, the primitive is a set of possible production vectors, say, T; Hicks's transformation surface is the efficient boundary of T, which, for convex production possibility sets, is somewhat more general, since differentiability is not required. In the notation introduced by Hicks and since universal, inputs are designated by negative entries, outputs by positive ones. Commodities are dated, so that production activities over time, with, for example, inputs preceding outputs, are included with no change in notation.

The transformation surface is assumed known with certainty in the V&C model. However, in the spot economy, future prices are not known with certainty. More generally, then, we assume that the vector of prices, p, is a random variable. Some components, however, may be known, so that we do not require that the support of the distribution of p have full dimensionality.

For any given choice, y, of production vector on the transformation surface, the profits, P, are given by,

$$P = p'y. \tag{1}$$

V&C assumes that, in some form, the firm is risk averse. In expected-utility theory, the notion of risk aversion is expressed by assuming a concave increasing utility function, U(P), such that the firm's behavior is described by the instruction,

$$\text{Max! } E[U(P)], \tag{2}$$

where E denotes mathematical expectation, subject to the condition that y is on the transformation surface,

$$T(y) = 0. \tag{3}$$

In the special case of risk-neutrality, the maximand, (2), becomes E(P), and the firm's decision problem is the same as if p were known with certainty to have the value, E(P). Hence, in this case, the expected value of the price vector is the CE price vector. For simplicity, I will also use the notation, p_m, for E(p), and similarly for the expected value of other variables.

In the general case of risk-aversion, the strategy for defining CEs is simple enough. Under the assumption that prices are all positive with positive probability, it is clear that the firm will never choose a technologically inefficient vector, since any feasible vector which dominates it will yield at least as much profit for all realizations of the random price vector and strictly more with positive probability. Hence, whatever production vector is chosen to maximize expected utility of profit will be technologically efficient and therefore will have a supporting price vector. By definition, this vector , if believed with certainty, would induce the same choice of production vector. Call this property quantity-matching. It is to be noticed that the set of supporting prices is a ray, not a point, even if the transformation surface is differentiable; in general, it is a cone.

There is another intuitive requirement for a CE price vector: if the price of any particular commodity is known with certainty, then the CE price for that commodity should be that value. In accordance with Fishburn [1986], p. 1200, call this property certainty-matching[1]. This property, if it can be satisfied, will tie down the CE price vector to a particular point on a ray. It will now be easy to see that we can choose the supporting prices to satisfy the second condition also.

The first-order condition for the maximization of (2) subject to (3) and (1) is,

$$E[U'(P)\ p] = mT_y, \tag{4}$$

where T_y is the gradient of T and m is a Lagrange parameter.

If p_e is the CE price vector, the maximizer y for (2-3) should also maximize,

$$P_e = p_e'y, \tag{5}$$

subject to (3). For this problem, the first-order conditions are,

$$p_e = nT_y, \tag{6}$$

for some Lagrange multiplier n. Since, by quantity-matching, y is the same for both problems, T_y is the same, so that,

$$E[U'(P)\ p] = (m/n)\ p_e. \tag{7}$$

Now consider any commodity, i, whose price is known for certainty. The ith component of (7) is,

$$E[U'(P)\ p_i] = (m/n)\ p_{e,i}. \tag{8}$$

Since p_i is a constant, the left-hand side can be written,

$$E[U'(P)]\ p_i,$$

while certainty-matching implies, $p_{e,i} = p_i$. Therefore,

$$m/n = E[U'(P)], \tag{9}$$

a condition which suffices to insure mataching of all certain prices. From (7) and (9),

$$p_e = E[U'(P)\ p]/E[U'(P)]. \tag{10}$$

Can one address the relation between CE and expected prices? In this framework, we can derive only for a special case the V&C claim that for <u>each</u> commodity the certainty-equivalent price departs from the expected price in the "conservative" direction, i.e., downwards for sales and upward for purchases. But the basic intuition is correct in general. In (10), multiply both sides by,

$$E[U'(P)]\ y'.$$

Since $y'p = p'y = P$, we have,

$$E[U'(P)]\ P_e = E[U'(P)\ P], \tag{11}$$

where $P_e = p_e'y$, i.e., the CE profit. By a simple statistical identity,

$$E[U'(P)\,P] = E[U'(P)]\,p_m + \mathrm{cov}\,[U'(P), P].$$

But $U'(P)$ is a decreasing function of P, and therefore their covariance must be negative if P is a non-degenerate random variable (for P to be a degenerate random variable, it would have to be that all the chosen inputs and outputs have certain prices). From (11), then,

$$E[U'(P)]\,P_e < E[U'(P)]\,P_m,$$

so that,

$$P_e < P_m, \tag{12}$$

the CE profits are less than the expected profits. Note that if production takes place under constant returns, then $P_e = 0$, so that (12) simply states that, under uncertainty, expected profits have to be positive. We can also write (12) as,

$$(p_e - p_m)\,y < 0, \tag{13}$$

so that in some general sense CE prices are more likely to fall short of expected prices for inputs than for outputs. If there is just one uncertain price, say for commodity j, then, since $p_{e,i} = p_i = p_{m,i}$ for all $i \neq j$, (13) reduces to,

$$(p_{e,j} - p_{m,j})\,y_j < 0, \tag{14}$$

so that, indeed, CE price falls short of or exceeds expected price according as the commodity is an output or an input.

But no such generalization is possible when more than one price is uncertain. If there are uncertainties about both input and output prices, for example, then it is not incompatible with (13) that the CEs of all uncertain prices exceed their expected values; this possibility will be illustrated as part of the following example.

Assume the utility function for profits is quadratic. The example will, in addition to showing how CEs can be computed, indicate that there is nothing special about the quadratic case when uncertainty relates to prices. Without loss of generality, the coefficient of the quadratic terms can be taken as -1/2 and the constant term as 0.

$$U(P) = bP - (1/2)\, P^2, \tag{15}$$

so that,

$$U'(P) = b - P. \tag{16}$$

Assume that there is one input and one output, and the set of possible production vectors is the ray spanned by (1,-1). Let y = output; then, $P = (p_1 - p_2)y = py$, where we abbreviate, $p = p_1 - p_2$, in this case, a scalar. The first-order condition for the maximization of E[U(P)] with respect to y is,

$$E[U'(P)p] = 0,$$

or,

$$bp_m = E(p^2)y. \tag{17}$$

Clearly, and in accordance with (13), we must have $p_m > 0$, in order for production to take place. Since $p = p_1 - p_2$, we assume that $p_{m,1} > p_{m,2}$.

Since the production possibility set is a cone, profits would have to be zero for any observed prices, so that $p_e = 0$, or,

$$p_{e,1} = p_{e,2}. \tag{18}$$

To calculate this common value, we use the formula (10):

$$p_{e,1} = E[U'(P)\, p_1]/E[U'(P)]. \tag{19}$$

From (16) and (17),

$$U'(P) = b - py = b[1 - p_m p/E(p^2)],$$

so that,

$$E[U'(P)] = b \text{ var } (p)/E(p^2). \tag{20}$$

Remember that p_1 and p_2 have some joint distribution, from which the distribution and therefore the mean and variance of p can be derived. By similar substitutions, one can calculate, $E[U'(P)\, p_1]$, and therefore, from (18) and (19) derive,

$$p_{e,1} = p_{e,2}$$

$$= [p_{m,1}[E(p_2{}^2)-E(p_1p_2)]+p_{m,1}[E(p_1{}^2)- E(p_1p_2)]]/\text{var } (p). \quad (21)$$

From (21), we can calculate the differences between CE and expected prices for both commodities. After simplification, we find,

$$p_{e,1}- p_{m,1} = p_m\ [\text{cov } (p_1,\ p_2) - \text{var } (p_1)]/\text{var } (p), \quad (22)$$

$$p_{e,2}- p_{m,2} = p_m\ [\text{var } (p_2) - \text{cov } (p_1,\ p_2)]/\text{var } (p). \quad (23)$$

Note that if, for example, the input price, p_2, were known with certainty, then both its variance and its covariance with any other variable, such as p_1, would be zero, so that both the CE and the expected values would equal the actual value. We can also note that if,

$$\text{var } (p_2) > \text{cov } (p_1,\ p_2) > \text{var } (p_1),$$

then the CE price would be above the expected price for both input and output prices. However, if the prices are independent random variables, then their covariance is zero, and the V&C hypothesis is valid: the CE for the output price is below its expected value, with the reverse for the input price.

In fact, this is a generally true statement, though the hypothesis that prices are independent random variables is hardly attractive. I will therefore only sketch the proof.

Theorem. If commodity prices are independent random variables, then, in the V&C model of production choice under uncertainty,

$$(p_{e,i} - p_{m,i})\, y_i < 0,$$

for all commodities with uncertain prices.

Sketch of proof: For convenience, let $V = U'(P)$. Since $E(Vp) = E(V)\, E(p) + \text{cov}\,(V, p)$, it follows from (10) that,

$$p_e = p_m + [\text{cov}\,(V, p)/V_m]. \qquad (24)$$

Let p_{-i} denote all the prices other than the ith. From independence, the conditional expectation of p_i given p_{-i} is the same as $p_{m,i}$. The covariance term in the ith component in (24) can be calculated as the expectation over p_{-i} of the conditional expectation given p_{-i} of,

$$(V - V_m)(p_i - p_{m,i});$$

but by elementary formulas, the last is the same as the covariance of V and p_i conditional on p_{-i}. For fixed p_{-i}, however, P is an increasing function of p_i if i is an output, and therefore $V = U'(P)$ is a decreasing function of p_i. The conditional covariance is therefore negative for every realization of p_{-i}, and hence cov $(V, p_i) < 0$, so that,

$$p_{e,i} < p_{m,i},$$

if $y_i > 0$.

3. QUADRATIC UTILITIES AND CERTAINTY-EQUIVALENCE: A BRIEF NOTE

What special association is there between quadratic payoffs and certainty-equivalence? The correct statement is simple enough: Suppose X is a random variable and a an action (both variables can be vectors). If the payoff function is quadratic jointly in <u>both</u> variables, then the decision is the same as if the distribution of X were concentrated at its mean.

The canonical example is to choose a as a best estimate of X, in the sense of minimizing,

$$E[(X-a)^2].$$

Clearly, the optimal choice of a is E(X) and is the same for all distributions with the same mean, in particular, if that value will occur with certainty. However, choice of production with uncertain prices does not fall within this framework. If U(P) is quadratic, with P = p'y, the payoff as a function of p (the random variable) and y (the action vector) contains terms of the forms,

$$p_i^2 y_i^2 \text{ and } p_i p_j y_i y_j,$$

which are of the fourth degree jointly in the two vector variables.

Certainty equivalence with quadratic payoffs occurs when prices are given but there is uncertainty in output, and the technology is linear. When there are complete contingent markets , then the general equilibrium over time has the property of maximizing a suitably weighted sum of all utilities (with weights which depend on preferences, endowments, and technology). The contingent prices are derived as Lagrange multipliers in the maximization (e.g., Hansen and Sargent [1988]). In this case, there is a sense in which certainty equivalence holds; that is, the production (or consumption) decision about <u>current</u> inputs or outputs is made replacing future prices by their expected value. But there is a significant difference from the V&C model; production decisions are flexible in the sense of the following section. The same remark applies to another frequently-cited example of certainty equivalence, the case of an exponential utility function with normally distributed random variables (see, e.g., Chow [1975, pp. 197-201]).

But in the V&C spot model and, more generally, when markets are incomplete, then prices are uncertain to the producer, and neither quadratic nor exponential utility functions imply any especially simple formulas for CE prices.

4. FLEXIBILITY IN PRODUCTION: DEFINITIONS AND EXAMPLES

Thus far, there is no underlying difficulty with the concept of CEs. It clearly is not as revealing as maximizing expected utility, since the level of activity is not derived; but it provides a correct description of choices under the assumptions made thus far.

The key assumption made thus far is that the entire production plan is chosen at one time. It overlooks the possibility of partial commitment. For example, a firm may choose some inputs now; tomorrow, it may choose among different outputs or it may choose both additional inputs and outputs. This is clearly true of farming, where certain inputs are chosen at planting time (seed, labor, and complementary capital) and further decisions are made at harvest time (labor and capital inputs, which, in turn, govern output). In particular, an interesting option is simply to abandon the production activity, by putting in zero inputs tomorrow and getting zero output. This may be profitable if tomorrow's output prices turn out to be sufficiently low relative to tomorrow's input prices.

The concept of flexible production was introduced to economic theory in an important though neglected paper of A.G. Hart [1942]. For a recent study, though with somewhat different timing assumptions, see Lau and Ma [1987]. In modern general equilibrium theory, flexibility of production is assumed in the notation, since in effect choices are made each period subject to the constraints imposed by past production decisions.

To formalize the notion of flexibility in the simplest possible way, I will assume that any production plan has only two periods, n(ow) and t(hen). At time n, the prices of some commodities are known and represented by the subvector, p_n. The prices of the remaining commodities, p_t, are random variables with a known joint distribution. At time n, decision is made as to the quantities of the corresponding commodities, y_n. At time t, the prices p_t become known, and the firm decides on the quantities of the remaining commodities, y_t, subject, of course, to the constraint that the entire production plan, (y_n, y_t) is feasible.

The CE prices should have the property of supporting the production plan. But what is the relevant production plan? There are two candidates. One is the production plan finally adopted, (y_n, y_t). The corresponding supporting prices will be called the <u>ex post</u> certainty equivalents and denoted by p_p. Alternatively, we can consider the prices that would support the choices, y_n, made in the first period; more precisely, we define <u>ex ante</u> certainty equivalent prices, p_a, as those that support (y_n, $\underline{y}_t$) for <u>some</u> feasible production plan.

To further bring out the essential characteristics of price flexibility, I assume that the firm is risk neutral. The striking implication of production flexibility is that even under risk neutrality uncertainty plays a decisive role.

Consider for a first example a fixed coefficient production process, with two inputs and one output. One input, denoted by K, is chosen at time n at a known price, r. At time t, the firm chooses a second input, denoted by L at price w, and simultaneously produces output Y = min (K, L) sold at price p. Prices w and p are of course not known at time n.

Suppose K has been chosen at time n. At time t, then, the firm will earn (p-w)L by choice of L, subject to the constraint $0 =< L =< K$. It will choose $L = K$ if $p > w$, $= 0$ if $p < w$ (assume the joint distribution of p and w to be continuous, so that the event, $p = w$, has probability 0). In the second case, the production vector is (-1, 0, 0), <u>which is technologically inefficient</u> and therefore has <u>no</u> supporting prices. Hence, <u>ex post certainty equivalent prices are not always defined</u>.

If $p > w$, then the production vector is (-1, 1, 1), which is supported by any triple (r_p, w_p, p_p) such that $p_p - w_p = r_p$. Since r is known, we take r_p =

r. The indeterminacy of the certainty equivalents fc.` p and w is here solely due to the non-differentiability of the production function.

Net revenue in the second period, after optimization, is $(p-w)K$ if $p > w$, 0 otherwise. For any number x, define x^+ to be max (x, 0). Then net revenue at time t is $(p-w)^+ K$. The firm's expected profit at time n is, then,

$$[E[(p-w)^+] - r] K,$$

and therefore equilibrium requires that,

$$r = E[(p-w)^+]. \tag{25}$$

For the ex ante CEs, we again require that, since r is known, $r_a = r$. Again, any pair w_a, p_a, such that, $p_a - w_a = r$ are ex ante certainty equivalents.

The coincidence of the ex ante and ex post CEs (when the latter exist) in this case does not generalize when the ex post choices are broader. To give another illustration, which will give some different perspectives, suppose the same inputs and outputs, with the same time structure, but suppose that,

$$Y = (K^{-1} + L^{-1})^{-1}. \tag{26}$$

Then, the marginal productivity of labor is given by,

$$MPL = [(L/K) + 1]^{-2}, \tag{27}$$

which is 1 at L = 0 and of course decreases in L for fixed K. It follows again that if $p < w$, the optimal policy at time t is to buy zero labor and therefore have zero output. On the other hand, if $p > w$, the firm sets,

$$L = [(p^{1/2} - w^{1/2})/w^{1/2}]\ K,\ y = [(p^{1/2} - w^{1/2})/p^{1/2}]\ K. \tag{28}$$

Net revenue at time t is,

$$py - wL = (p^{1/2} - w^{1/2})^2\ K \text{ if } p > w, \tag{29}$$

so that expected profits at time n are,

$$-\ rK + E[(p^{1/2} - w^{1/2})^{+2}]\ K,$$

and equilibrium requires that,

$$r = E[(p^{1/2} - w^{1/2})^{+2}]. \tag{30}$$

By certainty matching, the ex ante CEs for p and w would be a pair that satisfies (30) when the entire probability mass is concentrated at that point, i.e., p_a and w_a such that,

$$r = (p_a{}^{1/2} - w_a{}^{1/2})^2, \text{ with } r_a = r, \tag{31}$$

the last by the condition of certainty-matching. The indeterminateness of p_a and w_a follows from the indeterminateness of the production vector supported by the triple (r_a, w_a, p_a), since it can be <u>any</u> production vector with the given value of K.

Clearly, if $p < w$, there is no ex post CE, since $L = Y = 0$. If $p > w$ and MPK denotes the marginal productivity of capital, then the ex post certainty equivalents are those defined by the relations,

$$MPL = w_p/p_p,\ MPK = r_p/p_p.$$

By certainty-matching, $r_p = r$. From (2), $w_p/p_p = w/p$. If we substitute the actual values of K and L into the second relation and solve, we find that,

$$w_p = wr(p^{1/2} - w^{1/2})^2/p^2, \quad p_p = r(p^{1/2} - w^{1/2})^2/p. \tag{32}$$

These examples illustrate the ways in which ex ante and ex post CEs can be computed. The conditions for the existence of ex post CEs become problematic. There are examples where they always exist. Generalize the foregoing example to a general production function, $Y = F(K, L)$, concave and homogeneous of degree one, with the same time structure of choice and uncertainty resolution. Then, if MPL is infinite at $L = 0$ for $K > 0$, the ex post choice of L will be positive for any (positive) realizations of w and p. In this simple case, any choice will be efficient. We can easily determine the ex post CEs. First,

$$MPL = w/p = w_p/p_p, \tag{33}$$

the first by the timing of choice of L, the second by quantity-matching, since the ex post CEs must support the actual choice of L. Quantity-matching also implies that,

$$MPK = r_p/p_p;$$

but, as already seen, $r_p = r$, by certainty-matching, while MPK is a function of the capital-labor ratio and therefore of w/p. Hence, p_p is determined as a function of r and w/p, and then w_p is determined as a function of the same variables by (33).

However, it is easy to produce examples where there is positive production in the second period and yet there is no ex post certainty-equivalent price vector. Suppose the firm has two plants operating in different economies with different capital costs and wage rates, but the products are identical and sold on the same market. That is,

$$Y = F(K_1, L_1) + G(K_2, L_2).$$

Each of the production functions F and G is concave and homogeneous of degree one. There are now five prices, r_1, r_2. w_1, w_2, and p. Then by the reasoning just given applied to each plant separately, p_p is a function of r_1 and w_1/p and also a function of r_2 and w_2/p. Since w_1 and w_2 can vary separately, this is impossible.

5. OPTIMAL PRODUCTION CHOICE WITH FLEXIBILITY

I will now state more formally the optimal choice of a flexible probudction plan and follow this with the requirements for ex post and ex ante CEs in suitable generality. Let,

$$G(y_n, p_t) = \max_t p_t' y_t, \tag{35}$$

where, "$\max_t$" means the maximum over the set of subvectors y_t such that the entire production vector, (y_n, y_t) belongs to T, the production possibility set. Since total revenue is $p_n'y_n + p_t'y_t$, the vector, y_n, is chosen to maximize the expected value of,

$$p_n'y_n + E[G(y_n, p_t)]. \tag{36}$$

To solve the maximization problem in (35), introduce the Lagrangian,

$$L(y_t/y_n, p_t) = p_t'y_t + mT(y_n, y_t). \tag{37}$$

The first-order condition is,

$$p_t + mT_t = 0. \tag{38}$$

By the Envelope Theorem, the effect of a change in y_n on the maximum, G, is given by,

$$\partial G/\partial y_n = \partial L/\partial y_n = mT_n. \tag{39}$$

Maximizing (36) with respect to y_n then requires,

$$p_n + E(mT_n) = 0. \tag{40}$$

The optimal solution consists of a choice of y_n and a choice of y_t as a function of p_t (when it becomes known); it is characterized by (38), (40), and the transformation condition, $T(y_n, y_t) = 0$.

6. EX POST CERTAINTY EQUIVALENT PRICES

Given the optimal solution, under what conditions do there exist ex post and ex ante CE prices? First, let us remark on the possibility of ex post CEs. It appears that when there is effectively more than one commodity chosen in period n, it is unlikely that there exist ex post CE prices. If they did exist, the marginal rates of transformation among the commodities produced at time n would equal the ratios of their CE prices which, in turn, equal the actual prices, by certainty-matching. By the timing of choice, the marginal rates of transformation among the commodities chosen at time t equal the ratios of the actual prices. Therefore the CE prices at time t would be proportional to the actual prices. Hence,

$$p_p = (p_n, qp_t), \tag{41}$$

for some scalar q. By definition of ex post CE,

$$p_n + rT_n = 0, \tag{42a}$$

$$qp_t + rT_t = 0, \tag{42b}$$

for some r. By comparison of (42b) and (38), it is seen that r =qm. From (42a) and (40),

$$E(mT_n) + qmT_n = 0. \tag{43}$$

If there is only one commodity in the vector y_n (one input at the original time), then (43) can certainly be satisfied by suitable choice of q. If y_n has more than one component, then in general no solution exists. More specifically, if i and j are two commodities chosen now, we see that for (43) to hold,

$$T_i/T_j = E(mT_i)/E(mT_j), \tag{44}$$

so that the marginal rate of substitution between two commodities i and j, both chosen now, is not a random variable. Typically, this will be true if the commodities "now" are separable in prroduction from those chosen later, i.e., if we can find functions T*, U such that,

$$T(y_n, y_t) = T^*[U(y_n), y_t].$$

Conversely, if T_i/T_j is not a random variable, presumably because this ratio does not depend on input-output decisions "then," it follows that,

$$E(mT_i) = E[(T_i/T_j)\ mT_j] = (T_i/T_j)\ E(mT_j)],$$

so that (43) holds. We can say, roughly, that <u>ex post CE prices exist if and only if commodities "now" are aggregable into a single commodity independently of commodity choices made "then."</u>

7. EX ANTE CERTAINTY EQUIVALENT PRICES

Let me turn to the ex ante concept of CE. Given y_n as determined from the optimal solution, the question is, do there exist $\underline{y}_t$, $p_{a,t}$, and r satisfying,

$$p_n + rT_n(y_n, \underline{y}_t) = 0, \tag{45a}$$

$$p_{a,t} + rT_t(y_n, \underline{y}_t) = 0, \tag{45b}$$

and, of course, feasibility, i.e., $T(y_n, \underline{y}_t) = 0$. If #n and #t are the numbers of commodities chosen "now" and "then," respectively, there are #n +#t+ 1 equations and 2#t + 1 unknowns. Hence, we would expect that there will be an ex ante CE in general only if #t $\geqslant$ #n.

However, there is one special case which shows that this conclusion is not entirely correct, namely, when #t = 1, that is, only one commodity level is chosen at time t. But then, since $T(y_n, \underline{y}_t) = 0$, $\underline{y}_t$ is determined by y_n and is not a random variable. Therefore T_n and T_t are independent of p_t (now a scalar). Equation (38) still holds, so that,

$$m = -p_t/T_t,$$

and therefore,

$$E(m) = -p_{m,t}/T_t, \tag{46}$$

where $p_{m,t}$, it will be recalled, is the expected value of p_t.

Since T_n is not a random variable, (40) implies,

$$p_n + E(m)\, T_n = 0. \tag{47}$$

If we now set $r = E(m)$, then, from (47) and (46),

$$p_n + rT_n = 0,\ p_{m,t} + rT_t = 0,$$

so that the vector $(p_n, p_{m,t})$ form an ex ante CE (note that certainty matching is satisfied in the strongest sense of footnote 1).

If the "then" commodities are separable in production from the "now" commodities, then the effect is the same as if there were only one "then" commodity. The existence of ex ante CEs in this case follows immediately from a result already developed by Epstein [1980, Theorem 2, p. 978].

I now give an example in which $2 \leqslant \#t < \#n$ and show, in agreement with the earlier conjecture, that no ex ante CE exists.

Suppose there are three possible activities, defined by Cobb-Douglas production functions with different exponents. The capital goods, chosen in period n, are different commodities. In period t, homogeneous labor is purchased at wage w and allocated among the three activities. The total output is then sold at a price p. It will be seen that under uncertainty as to w and p it may well be desirable to invest in all three activities, yet under any known w and p there will be investment in only two activities. (The point can be seen even more clearly with three linear activities, but it might be felt that non-smoothness of the production possibility set was the cause.)

For any Cobb-Douglas production function, second-period revenue at optimal choice of labor is proportional to,

$$p^{1+b} w^{-b},$$

per unit capital, for some $b > 0$. Choose the proportionality constants all equal to 1, and use $b = 0$, $b = 1$, and $b = 2$ for the three activities. Let r_b ($b = 0, 1, 2$) be the price of the capital good used in activity b. Then for an optimal choice under uncertainty, all three activities could have positive capital goods if the equilibrium conditions,

$$r_b = E(p^{1+b} w^{-b}),\ b = 0, 1, 2, \tag{48}$$

are satisfied. If ex ante certainty equivalents p_a and w_a existed, then it would be necessary that,

$$r_b = p_a^{1+b} w_a^{-b},\ b = 0, 1, 2. \tag{49}$$

But (49) implies that,

$$r_0 r_2 = r_1^2. \tag{50}$$

If p and w were both known with certainty, then indeed (48) would imply the satisfaction of (50). But if p and w (or even one of them) were unknown, then (50) would hold only by accident. To illustrate, suppose that the joint distribution of p and w satisfy the (not unreasonable) condition that,

$$p = wu,$$

where u is a random variable independent of w. Then, substitution into (39) shows that,

$$r_b = E(wu^{1+b}) = E(w)\,E(u^{1+b}). \tag{51}$$

If we define m_k to be the kth moment of the random variable u,

$$m_k = E(u^k),$$

then (50) reduces to,

$$m_1 m_3 = m_2^2. \tag{52}$$

This condition is not satisfied for any distribution of a positive random variable. To see this, let f(u) be the density of u. Then,

$$M = m_1 m_3 - m_2^2 = (\int u\ f(u)\ du)(\int v^3 f(v)\ dv) - (\int u^2 f(u)\ du)\ (\int v^2 f(v)\ dv)$$

$$= \iint (uv^3 - u^2v^2)\ f(u)\ f(v)\ du\ dv = \iint u^2v^2(u^{-1}v - 1)\ du\ dv.$$

If we interchange u and v, we get again the same M; if the two expressions for M are averaged, we find that,

$$M = (1/2) \iint u^2v^2(u^{-1}v + uv^{-1} - 2)\ f(u)\ f(v)\ du\ dv.$$

Let $z = u^{-1}v$; then,

$$u^{-1}v + uv^{-1} = z + z^{-1} - 2 \geqslant 0 \text{ for all } z > 0,$$

with the strict inequality for $z \neq 1$. If u has a non-degenerate distribution, then u = v with probability less than 1 (in fact, 0 for a continuous distribution) and therefore,

$$u^{-1}v + uv^{-1} - 2 > 0,$$

with positive probability. Hence, $M > 0$, so that (52) cannot hold.

A variation of this example also shows that even if (45a-b) are satisfied, the resulting prices need not satisfy certainty-matching. It may well be that the firm is certain about the price of the ith commodity in period t, and certainty-matching demands that $p_{a,i} = p_i$. Suppose just two of the activities of the previous example exist, those with $b = 1$ and $b = 2$. Then, p_a and w_a are determined by the relations,

$$E(p^2w^{-1}) = p_a^2 w_a^{-1},$$

$$E(p^3w^{-2}) = p_a^3 w_a^{-2}.$$

Suppose that p is known for certain, so that the left-hand sides of the equations can be written,

$$p^2E(w^{-1}) \text{ and } p^3E(w^{-2}),$$

respectively. Square both sides of the first equation and divide by the second.

$$p_a = [[E(w^{-1})]^2/E(w^{-2})]\ p.$$

But $E(w^{-2}) = \text{var}\ (w^{-1}) + [E(w^{-1})]^2$, so that $p_a < p$, necessarily, and therefore certainty-matching is violated.

Hence, it is reasonable to conclude that ex ante CEs exist only when there is more choice at the second stage than at the first.

8. CONCLUSIONS

To sum up, in the absence of production flexibility, certainty-equivalent (CE) prices, satisfying both quantity-matching and certainty-matching, exist, i.e., they support the same choice of production vector as the original uncertainty and any price known for certain will have that value as CE.

If there is flexibility in production, so that choices made later tcan take account of additional information, then there are two possible definitions of CE. The ex post definition however rarely exists. The ex ante CE prices do not exist if there is insufficient flexibility and probably exists whenever there is enough.

NOTES

[1] One might demand even more strongly that, if any linear combination of prices is known for certain, the same linear combination of CE prices has the known value. A still stronger and more elegant statement of certainty-matching is that the CE price vector always lies in the convex hull of the support of the distribution of p. Even this last condition is satisfied the following construction.

[2] As promised, this definition satisfies the strongerst version of certainty-matching defined in footnote 1. If f(p) is the density of p, let g(p) = U'(P) f(p)/E[U'(P)]. This is also a density (a non-negative function with integral equal to 1) and is non-zero just on the same set where f(p) is non-zero. From (10), p_e would be the mean of p if the density were g(p) and therefore certainly lies in the convex hull of the support of p.

REFERENCES

Allais, M. (1953) 'Generalisation des theories de l'equilibre economique general et du rendement social au cas du risque', Econometrie (Paris: Editions du Centre National de la Recherche Scientifique) Pp. 81-109.

Arrow, K.J. (1953) 'Role des valeurs boursieres pour la repartition la meilleure des risques', Econometrie (Paris: Editions du Centre National de La Recherche Scientifique) Pp. 41-47.

Bernoulli, D. (1738) 'Specimen theoriae novae de mensura sortis', Commentarii academiae scientiarum imperiales Petropolitanae 5: 175-192.

Chow, G. (1975) Analysis and Control of Dynamic Economic Systems (New York: Wiley.

Epstein, L. (1980) 'Multivariate risk independence and functional forms for preferences and technologies', Econometrica 48: 973-985.

Fishburn, P. (1986) 'Implicit mean value and certainty equivalence' Econometrica 1197-1205.

Hansen, L.P, and T.J. Sargent (1988) 'Linear models of economic time series', Draft manuscript.

Hart, A.G. (1942) 'Risk, uncertainty, and the unprofitability of compounding probabilities' O. Lange, F. McIntyre, and T.O. Yntema (eds.) Studies in Mathematical Economics and Econometrics (Chicago: University of Chicago Press) pp. 110-118.

Hicks, J.R. (1931) 'The theory of uncertainty and profit' Economica 11: 170-189

Hicks, J.R. (1935) 'A suggestion for simplifying the theory of money' Economica.

Hicks, J.R. (1939) Value and Capital (Oxford: Oxford University Press) Second Edition, 1946.

Hicks, J.R. (1962) 'Liquidity' Economic Journal 72: 187-202.

Jevons, H.S. (1957) The Theory of Political Economy, Fifth Edition, (Reprinted 1965, New York: Augustus M. Kelley)

Lau, L.J., and B.K. Ma (1987) 'Certainty equivalence in expected profit maximization implies impossibility of loss' Economics Letters 25: 27-130.

Markowitz, H. (1952) 'Portfolio selection' Journal of Finance 7: 77-91

Marschak, J. (1938) 'Money and the theory of assets' Econometrica 6: 311-325

Marschak, J. (1949) 'Role of liquidity under complete and incomplete information' American Economic Review Papers and Proceedings 39(May): 182-195.

Marshall, A. (1948) Principles of Economics, Eighth Edition. (London: Macmillan)

von Neumann, J., and O. Morgenstern (1947) Theory of Games and Economic Behavior, Second Edition (Princeton, N.J.: Princeton University Press)

Ramsey, F.P. (1931) 'Truth and probability' F.P. Ramsey, The Foundations of Mathematics and Other Logical Essays (London: K. Paul, Trench, Trubner)

Theil, H. (1954) 'Econometric models and welfare maximisation' Weltwirtschaftliches Archiv 72: 60-83